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LETTER TO THE EDITOR

# An integrability test for differential-difference systems

A Ramani† B Grammaticos‡ and K M Tamizhmani§

† CPT, Ecole Polytechnique, 91128 Palaiseau, France
‡ LPN, Université Paris VII, Tour 24-14, 5ème étage, 75251 Paris, France
§ Department of Mathematics, Pondicherry University, Kalapet, Pondicherry 605104, India

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**Abstract.** We present a novel integrability criterion that is adapted to differential-difference systems. It is based on the recently introduced notion of singularity confinement and uses elements from the singularity analysis approach, which has proved to be a most successful integrability criterion for continuous systems. A study of the Toda lattice illustrates the applicability of our method.

The singularity analysis [1], also known under the name of the Painlevé method, has been a most useful tool for the investigation of the integrability of differential systems over the past 15 years. Hundreds of systems have been studied, leading to the discovery of a host of interesting new integrable cases. However, the essence of the method, i.e. the local study of the analytic properties of the solutions, did not lend itself to generalizations so as to cover the case of discrete systems. The decisive step in this direction was taken with the introduction of the singularity confinement method [2]. It is based on the fact that singularities that appear accidentally (due to the choice of initial conditions) do not propagate *ad infinitum* in the case of integrable discrete systems with the evolution of the mapping, but disappear after a few iterations. This method is essentially non-local and thus does not lend itself to the study of continuous systems. So, the question has remained open as to how one investigates the integrability of systems that are both discrete and continuous, i.e. differential-difference equations. The physical examples of such systems abound. Any classical interacting $N$-particle ensemble can be represented by a differential- (in the continuous time) difference (in the particle label) system of equations (the dependent variable being the particle position). The aim of the present letter is to present a method that makes possible the treatment of these elusive discrete-continuous cases.

Let us, just to fix the ideas, present in a nutshell the principle of the continuous singularity analysis and of the discrete singularity confinement. Given a nonlinear ordinary differential equation (with some restrictions as to its form) one looks for the singular solutions around a movable (i.e. initial condition-dependent) singularity. If for every such singularity the expansion of the solution is of Laurent series type then the equation is said to possess the Painlevé property and, according to the Ablowitz–Ramani–Segur conjecture [3], it should be integrable. In the case of discrete systems, we consider a recursion relation allowing us to compute a given term from the knowledge of the preceding ones. We look for all the possible singularities of this recursion leading to a divergent term. If, in the course of the subsequent iterations, this divergence disappears, the mapping is said to possess confined singularities and according to the Grammaticos–Ramani–Papageorgiou conjecture [2] it must be integrable.

In what follows we will try to blend the two methods on the classical paradigm of the Toda lattice [4]. As is well known, this system represents an ensemble of particles on a line interacting via an exponential interaction between nearest neighbours. The equations of motion lead to

$$\ddot{x}_n = e^{x_{n+1}-x_n} - e^{x_n - x_{n-1}} . \tag{1}$$

This system is traditionally transcribed into a purely algebraic form through the transformation $a_n = e^{x_{n+1}-x_n}$, $b_n = \dot{x}_n$, leading to

$$\dot{a}_n = a_n(b_{n+1} - b_n) \tag{2a}$$

$$\dot{b}_n = a_n - a_{n-1} . \tag{2b}$$

The classical way of performing the singularity analysis of this system is by imposing some periodicity condition, i.e. we assume that we have a chain of $N$ particles such that $x_0 = x_N$. The singularity analysis is then performed for a given number of particles (usually small). However for a general $N$ it is usually very difficult to perform the singularity analysis even for some given simple singular behaviour (although in the case of the Toda lattice this can be done for the simplest leading singularity [5]). Moreover with increasing $N$ new singular behaviours appear [6] and thus one cannot make an exhaustive study. The main drawback of this approach is that the small $N$ cases are far from being generic and thus one cannot draw firm conclusions based on the study of such cases alone.

The spirit of the singularity confinement is quite different. One looks for the spontaneous appearance of a singularity for some $n$ (when the particle number is interpreted as the number of steps in some recursion). Thus one does not study the solutions that are allowed to be singular for every $n$ but only those that *become* singular at some $n$. In this context relation (2) is to be interpreted as

$$a_n = a_{n-1} + \dot{b}_n \tag{3a}$$

$$b_{n+1} = b_n + \frac{\dot{a}_n}{a_n} . \tag{3b}$$

We start by assuming that both $b_n$ and $a_n$ are non-divergent and that the singularity appears in step $n+1$. In fact, due to the presence of the logarithmic derivative in (3*b*), a pole may appear in $b_{n+1}$ if $a_n$ vanishes at some time $\tau = t - t_0$. Let us start with the simplest case of a single zero, i.e.

$$a_n = \alpha\tau$$

where $\alpha = \alpha(t)$ and $\alpha(t_0) \neq 0$. Substituting in (3*b*) we find

$$b_{n+1} = \frac{1}{\tau} + b_n + \frac{\dot{\alpha}}{\alpha}$$

$$a_{n+1} = -\frac{1}{\tau^2} + \dot{b}_n + \frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \alpha\tau .$$

Iterating further we obtain

$$b_{n+2} = -\frac{1}{\tau} + b_n + \frac{\dot{\alpha}}{\alpha} - 2\tau\left[\dot{b}_n + \frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha}\right)^2\right] - A\tau^2 + \mathrm{O}(\tau^3)$$

$$a_{n+2} = (4A - 7\alpha)\tau + \mathrm{O}(\tau^2)$$

where $A$ is a quantity depending on $\alpha$ and $b_n$. Iterating further we obtain a finite result for $b_{n+3}$. Thus the singularity that appeared at $b_{n+1}$ due to the simple root in $a_n$ is confined after two steps.

The preceding analysis shows us that the *integrable* Toda lattice has indeed confined singularities (at least of the $a_n = \alpha\tau$ type). Still a much more stringent test of the singularity confinement method would be to be able to single out this system amid a larger family of *non-integrable* ones. To this end we introduce a (minimal) change to the equations of motion (3) in the form

$$a_n = \lambda a_{n-1} + \dot{b}_n \tag{4a}$$

$$b_{n+1} = \mu b_n + \frac{\dot{a}_n}{a_n} \tag{4b}$$

and repeat the analysis of the previous paragraph. Starting from the same assumption of finite $b_n$ and $a_n = \alpha\tau$ we find that $a_{n+2}$ diverges *unless* we have $\lambda + \mu = 2$. Using this constraint (that suffices in order to obtain a vanishing $a_{n+2}$) we compute $b_{n+3}$. The latter is divergent unless $\lambda = \mu = 1$ and thus among all possible systems of the form (4) only the Toda ($\lambda = \mu = 1$) passes the integrability test.

Clearly the vanishing $a_n$ behaviour examined above and which induces the divergence of $b_{n+1}$ is not the only one. One can imagine higher-order zeros of the type $a_n = \alpha\tau^k$. Depending on the value of $k$, more and more intermediate steps will be necessary for the confinement of the singularity. If, for example, we start with a regular $b_n$ and $a_n = \alpha\tau^2$ we obtain $b_{n+1} \propto 2/\tau$ and $a_{n+1} \propto -2/\tau^2$. At the next iteration we find a finite $b_{n+2}$ but $a_{n+2}$ diverges as $-2/\tau^2$, leading to $b_{n+3} \propto -2/\tau$ and $a_{n+3} \propto \tau^2$. At this step the singularity is confined and we obtain a finite $b_{n+4}$. Compared to the case $a_n \propto \tau$ one more step was needed here. Thus it is in principle impossible to treat the general case of $a_n \propto \tau^k$. This is not unlike the classical Painlevé-type treatment for differential-difference systems (through periodification) where one cannot, in general, assess every possible singularity. However, the main advantage of the present method is that the simplest singular behaviour is also the most generic one. Its study yields the most important integrability constraints for the system. Thus, by studying just a few of the simplest such singular behaviours one can obtain enough constraints in order to determine the 'integrability candidate' subcases of a system containing free parameters.

Summarizing our presentation, we can affirm that a new method seems to emerge from the combination of the powerful integrability analysis and confinement. It is based on the confinement of the movable singularities that can appear in continuous systems at some lattice point. It presents definite advantages over the conventional Painlevé method because it singles out the most generic singularities of the system and studies their consequences. From the treatment of the Toda lattice we can conclude that the method can be applied algorithmically in an efficient way. It would be interesting to implement it to more general systems and, perhaps, to discover new cases of integrability.

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